Exactly solvable path integral for open cavities in terms of quasinormal modes

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We evaluate the finite-temperature Euclidean phase-space path integral for a scalar field in a leaky cavity. If the source is confined to the cavity, after integrating out the environment one can expand the ensuing effective cavity action in terms of the *quasinormal modes* (QNMs)—the exact, damped eigenstates of the classical evolution operator, known to be complete for a large class of models. Dissipation makes the effective-action matrix nondiagonal in the QNM basis. Its inversion in the Gaussian path integral for the generating functional thus is nontrivial, but feasible using a novel QNM sum rule. The results are consistent with those of canonical quantization.

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I. INTRODUCTION

Open systems have been amply studied both in classical and quantum physics: e.g., optics—cavity QED [1] applicable to laser physics [2] or microdroplets, the spherical analog—and solid-state physics, where Josephson [3] and Kondo phenomena [4], etc., all allow a ''system-bath'' description. The concept is also relevant to acoustics (e.g., sound emanating from musical instruments) and on a very different scale to gravitational astrophysics [5].

In several papers, we have studied open wave systems (Eq. (2.1) with a nontrivial mass density $\rho(x)$ below or, equivalently [6], the Klein-Gordon equation $[\partial_t^2-\partial_x^2+V(x)]\psi=0$). The waves propagate in a ''universe'': an open ''cavity'' plus an infinite ''outside'' (the bath). Dissipation is caused by leakage from the former to the latter [7]. Under conditions specified later, the discrete set of cavity resonances or *quasinormal modes* (QNMs)—exponentially decaying eigensolutions of the (non-Hermitian) evolution operator—is complete in the cavity and hence can be used for exact expansions. This eliminates the outside from the description, and one no longer has to deal with the dense set of modes of the universe (MU). In terms of these QNMs, one can establish a formalism which closely parallels the usual one for conservative, Hermitian systems. Applications include perturbation theory and, of particular interest here, (canonical) second quantization [8,9]. For a review, see Ref. [10].

Of course, there are many other ways to eliminate the bath, leading to, e.g., Langevin and master equations [2,11]. Especially suited for open quantum systems is the path integral [4,12], which one first writes down for the generating functional (or density matrix) of the universe. The pertinent action follows from the Hamiltonian. One then performs the integral over the bath only, in which the system variables figure as constants. Since the bath is usually taken harmonic (in fact, for a meaningful separation into system plus bath one needs this or some other simplifying property), this can be done exactly. In the remaining path integral over the damped system variables, the *effective action* accounts for the environment in a way guaranteed to be consistent with quantum mechanics (i.e., not merely phenomenological). This is a convenient starting point for approximations, qualitative analysis, or numerics.

This paper synthesizes the QNM and path-integral approaches to open wave systems. In Sec. II we review the classical QNM series. In Sec. III we present the path integral; since our ultimate interest is in the cavity fields, the source is chosen to couple to those only, facilitating the elimination of the bath. The form of the ensuing effective action's damping term is still well known, and QNM expansion in Sec. IV combines the merits of an effective action with those of a discrete basis [10]. If not only the bath but also the cavity is harmonic, a sum rule, also derived, now enables the cavity integral over the expansion coefficients to be performed. (For nonlinear actions, this step is the starting point for perturbation theory; see Sec. VI.) Since the action is bilinear and the QNMs are not orthogonal in the usual sense, this means inverting a nondiagonal infinite matrix (in contrast to the bath integral, which can be done for each degree of freedom separately). The result agrees with that of canonical quantization: both yield the same correlators. While the systems in Refs. [3,4,12] typically have *few* degrees of freedom, only their baths being essentially infinite, this paper carries out the analogous program for a damped *field*. In Sec. V we consider a source coupling to the field but not to its momentum. Then, the action matrix *can* be made diagonal; this parallels the canonical approach. While the resulting formulas look simpler, some pitfalls are pointed out. Closing remarks are made in Sec. VI.

II. CLASSICAL FIELDS

For closed linear systems, eigenfunction expansions, based on the normal modes (NMs) of the evolution operator, are a vital tool. However, in open systems, any state will decay, so NMs do not exist. Consider the real scalar one-dimensional wave equation in a cavity $0\leqslant x\leqslant a$,

$$\rho(x)\partial_t^2\phi=\partial_x^2\phi, \tag{2.1}$$

with a node

$$\phi(x=0,t)=0 \tag{2.2}$$

at one end but with the outgoing-wave condition (OWC)

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$$\phi'(a^+,t) = -\dot{\phi}(a,t) \tag{2.3}$$

at the other. The OWC means that, just outside the cavity, $\phi(x,t)=\phi(x-t)$; it is stated at a^+, since for many models a singularity in $\rho(x=a)$ can cause a jump in $\phi'(x)$ [13]. Equation (2.3) renders the cavity leaky but not absorptive. The QNMs read $\phi(x,t)=f_j(x)e^{-i\omega_j t}$, with

$$[\partial_x^2+\rho(x)\omega_j^2]f_j=0, \quad f_j(0)=0, \quad f_j'(a^+)=i\omega_j f_j(a). \tag{2.4}$$

One easily verifies that $\mathrm{Im}\,\omega_j<0$, so $e^{-i\omega_j t}$ is indeed decaying. The frequencies ω_j, ordered according to increasing real parts, are spaced by $\Delta\omega\sim\pi/a$, roughly as for closed systems of size a. They occur in pairs $\omega_{-j}=-\omega_j^*$ (unless $\mathrm{Re}\,\omega_j=0$), and one can choose $f_{-j}=f_j^*$. While ϕ is real, the ω_j and f_j are complex, hence the pairing of modes.

Usually, eigenfunction expansions rely on Hermiticity of the evolution operator, which only holds for closed systems. One way out is to embed the cavity into a universe $0\leqslant x\leqslant\Lambda$ with a node at $x=\Lambda\to\infty$, and use the MU. Namely, Eqs. (2.1)–(2.3) are the restriction to $x\leqslant a$ of Eq. (2.1) on $0\leqslant x<\infty$, if $\rho(x>a)=1$ and $\phi'(x>a,t=0)=-\dot{\phi}(x>a,t=0)$. However, then one has to deal with a continuum ($\Delta\omega\sim\pi/\Lambda\to0$) instead of a discrete set of states in the closed case. Also, the self-contained Eqs. (2.1)–(2.3) show that even for a damped cavity the (thermo)dynamics can be studied *without* explicit reference to the outside, which is the main goal when second quantizing the open system.

A QNM expansion in terms of the cavity variables only, avoiding the disadvantages of the MU and exact for any amount of damping, *does* exist if (a) $\rho(x=a)$ has at least a step discontinuity, demarcating a cavity. (b) $\rho(x>a)=1$, so that the outside does not backscatter outgoing waves, enabling its complete elimination. See Ref. [10], and references therein.

First, one shows that the retarded Green function has the representation

$$G^{\mathrm{R}}(x,y;t)=\sum_j \frac{f_j(x)f_j(y)}{2i\omega_j}e^{-i\omega_j t} \quad (0\leqslant x,y\leqslant a, t>0), \tag{2.5}$$

where the f_j's are normalized as in Eq. (2.8) below. Thus, the dynamics is contained entirely in the QNMs. Second, since Eq. (2.1), as with any Hamiltonian problem, involves both position and momentum, one introduces *pairs* $\boldsymbol{\phi}=(\phi,\hat{\phi})^{\mathrm{T}}$ with $\hat{\phi}\equiv\rho\dot{\phi}$, so that $\boldsymbol{f}_j=(f_j,-i\rho\omega_j f_j)^{\mathrm{T}}$. The space of all outgoing-wave pairs, satisfying Eqs. (2.2) and (2.3), will be denoted as Γ [14].

Using these pairs, the time evolution generated by Eq. (2.5) can be recast in the form

$$\boldsymbol{\phi}(t)=\sum_j a_j(t)\boldsymbol{f}_j, \quad 2\omega_j a_j(t)=(\boldsymbol{f}_j,\boldsymbol{\phi}(t)), \tag{2.6}$$

with $a_j(t)=a_j(0)e^{-i\omega_j t}$ and the *bilinear map* for $\boldsymbol{\zeta},\boldsymbol{\chi}\in\Gamma$

$$(\boldsymbol{\zeta},\boldsymbol{\chi})=i\left\{\int_0^{a^+}dx\,[\zeta(x)\hat{\chi}(x)+\hat{\zeta}(x)\chi(x)]+\zeta(a)\chi(a)\right\}. \tag{2.7}$$

By letting $t\downarrow0$ in Eq. (2.6) one obtains a *two-component expansion* for any $\boldsymbol{\phi}\in\Gamma$, proving completeness of the QNMs. With

$$\mathcal{H}=i\begin{pmatrix}0 & 1/\rho\\ \partial_x^2 & 0\end{pmatrix},$$

Eq. (2.1) becomes $i\partial_t\boldsymbol{\phi}=\mathcal{H}\boldsymbol{\phi}$, in analogy with quantum mechanics. Equation (2.4) for f_j then reads $\mathcal{H}\boldsymbol{f}_j=\omega_j\boldsymbol{f}_j$. Even though the system is open, $\mathcal{H}$ is symmetric: $(\boldsymbol{\zeta},\mathcal{H}\boldsymbol{\chi})=(\mathcal{H}\boldsymbol{\zeta},\boldsymbol{\chi})$ for any $\boldsymbol{\zeta},\boldsymbol{\chi}\in\Gamma$. This yields ''orthogonality''

$$(\boldsymbol{f}_j,\boldsymbol{f}_k)=2\omega_j\delta_{jk} \tag{2.8}$$

by the usual proof, leading to uniqueness of the expansion [Eq. (2.6) for the first component alone would not be unique]. In Eq. (2.8), we have already implemented the normalization used in Eqs. (2.5),(2.6); in general the right-hand side (RHS) is not real, stressing the difference between Eq. (2.7) and a standard sesquilinear scalar product. The bilinearity of Eq. (2.8) also fixes the phase of f_j.

Instead of as an ''orthogonal'' expansion using a bilinear map, Eq. (2.6) can also be regarded as a biorthogonal expansion involving the standard inner product. This becomes useful when several QNMs merge [15]; we will only consider this briefly in Appendix A.

III. ELIMINATION OF THE OUTSIDE

We shall express the generating functional for the cavity field in terms of the QNMs. We want results for finite temperature, so a Euclidean formulation is advantageous. Since Eq. (2.6) involves two components, we must use a phase-space path integral

$$S\{\boldsymbol{\chi}\}=\left\langle \mathcal{T}_\tau \exp\left\{\int_0^\beta d\tau\,(\boldsymbol{\phi}(\tau),\boldsymbol{\chi}(\tau))\right\}\right\rangle \tag{3.1}$$

$$=Z^{-1}\int\mathcal{D}\boldsymbol{\phi}(x,\tau)\,\exp\left\{\int_0^\beta d\tau\left[(\boldsymbol{\phi},\boldsymbol{\chi})-\int_0^{a+\Lambda}dx\left(\frac{1}{2\rho}\hat{\phi}^2+\frac{1}{2}\phi'^2-i\hat{\phi}\dot{\phi}\right)\right]\right\}, \tag{3.2}$$

with $\mathcal{D}\boldsymbol{\phi}\equiv\mathcal{D}\phi\mathcal{D}\hat{\phi}$, $\Lambda\to\infty$, and $\beta=1/T$ ($\hbar=k_{\mathrm{B}}=1$). By Eq. (2.8), the form (2.7) of the coupling of the real [16] source $\boldsymbol{\chi}$ to the cavity field only (see Sec. I) will be convenient upon QNM expansion (2.6). Imaginary-time ordering $\mathcal{T}_\tau$ is needed in Eq. (3.1), where $\boldsymbol{\phi}$ is an operator, but not in the c-number formula (3.2). Below, the meaning of $\boldsymbol{\phi}$ will follow from the context. The normalization Z formally equals the path integral with $\boldsymbol{\chi}\mapsto0$; in fact both are infinite, only their ratio is meaningful. In the following we shall cancel all $\boldsymbol{\chi}$-independent factors against their counterparts in Z, without reflecting this in the notation. The boundary conditions are $\phi(0,\tau)=\phi(a+\Lambda,\tau)=0$ and, due to the trace implicit in the expectation (3.1), $\phi(x,0)=\phi(x,\beta)$. No conditions can be imposed on $\hat{\phi}$, which typically is completely discontinuous as the action does not contain $\hat{\phi}'$. This also means that phase-space path integrals can be tricky [17,18]. However, this in general matters only beyond the semiclassical approximation, which is exact for our linear problem [19].

Let us split the integral into a cavity and a bath factor: $Z^{-1}\int\mathcal{D}\boldsymbol{\phi}=Z_{\rm c}^{-1}\int\mathcal{D}\boldsymbol{\phi}_{\rm c}\, Z_{\rm b}^{-1}\int\mathcal{D}\boldsymbol{\phi}_{\rm b}$, where the latter runs over fields on $(a,a+\Lambda)$ with a given boundary value $\phi(a,\tau)$. The integral over bath momenta is trivial since it can be done for each space-time grid point separately; introducing $\xi\equiv x-a$ and using $\rho(x>a)=1$, one is left with $Z_{\rm b}^{-1}\int\mathcal{D}\phi_{\rm b}\exp\{-\mathcal{S}_{\rm b}\}$, where $\mathcal{S}_{\rm b}=\int_0^\beta d\tau\int_0^\Lambda d\xi\ \tfrac{1}{2}(\dot{\phi}^2+\phi'^2)$. Expanding $\phi_{\rm b}(\xi,\tau)=T\Sigma_m e^{-i\nu_m\tau}\{\phi_m(a)(\Lambda-\xi)/\Lambda+\Sigma_{u=1}^\infty\phi_{um}\sin(\pi u\xi/\Lambda)\}$ (with the Bose frequencies $\nu_m=2\pi mT$, $m\in\mathbf{Z}$), one has

$$\mathcal{S}_{\rm b}=\frac{T}{2}\sum_m\left\{\frac{\Lambda^2\nu_m^2+3}{3\Lambda}|\phi_m(a)|^2+\sum_{u=1}^\infty\left[\frac{\Lambda^2\nu_m^2+\pi^2u^2}{2\Lambda}|\phi_{um}|^2+\frac{2\Lambda\nu_m^2}{\pi u}\mathrm{Re}[\phi_{um}\phi_{-m}(a)]\right]\right\}$$

$$=\frac{T}{2}\sum_m\left\{\frac{\Lambda^2\nu_m^2+3}{3\Lambda}|\phi_m(a)|^2+\sum_{u=1}^\infty\left[\frac{\Lambda^2\nu_m^2+\pi^2u^2}{2\Lambda}|\overline{\phi}_{um}|^2-\frac{2\Lambda^3\nu_m^4|\phi_m(a)|^2}{\pi^2u^2(\Lambda^2\nu_m^2+\pi^2u^2)}\right]\right\},\qquad(3.3)$$

where $\overline{\phi}_{um}=\phi_{um}+2\nu_m^2\phi_m(a)/\{\pi u[\nu_m^2+(\pi u/\Lambda)^2]\}$. Changing to $\overline{\phi}_{um}$ does not alter the domain [especially not in a $\phi_m(a)$-dependent way] since both ϕ_{um} and $\overline{\phi}_{um}$ run over all $\mathbf{C}$, subject only to $\phi_{um}=\phi_{u,-m}^*$ and $\overline{\phi}_{um}=\overline{\phi}_{u,-m}^*$. Upon completing the square in Eq. (3.3), the integral thus yields a $\phi(a)$-independent constant which cancels against $Z_{\rm b}$:

$$\int\frac{\mathcal{D}\phi_{\rm b}}{Z_{\rm b}}\exp\{-\mathcal{S}_{\rm b}\}=\exp\left\{\frac{T}{2}\sum_m|\phi_m(a)|^2\left[-\frac{\Lambda^2\nu_m^2+3}{3\Lambda}+\sum_{u=1}^\infty\frac{2\Lambda^3\nu_m^4}{\pi^2u^2(\Lambda^2\nu_m^2+\pi^2u^2)}\right]\right\}$$

$$=\exp\left\{-T\sum_m\frac{1}{2}|\nu_m||\phi_m(a)|^2\right\}\quad\text{for}\quad\Lambda\to\infty,\qquad(3.4)$$

where to arrive at Eq. (3.4) we used $\Sigma_{u=1}^\infty u^{-2}(1+\epsilon^2u^2)^{-1}=\pi^2/6-\pi|\epsilon|/2+\mathcal{O}(\epsilon^2)$ for $\epsilon=\pi/\Lambda\nu_m$, leading to the cancellation of the $\mathcal{O}(\Lambda)$ terms in the exponent on the first line.

The Caldeira-Leggett type [4] exponent in Eq. (3.4) is the quantum finite-T equivalent of an Ohmic-damping term. Its emergence is expected, given the correspondence between our transmission-line environment [20] and the oscillator baths used originally [12]: if $\Lambda\to\infty$, waves escaping into the homogeneous outside string will never be scattered back, so that the outside acts as a sink. Since Eq. (2.1) is dispersionless, this damping is frequency independent. Classically this yields Eq. (2.3), where ϕ' is precisely the string tension; apparently, this force equals $-\dot{\phi}$. This velocity proportionality is reflected by the first power of ν_m in Eq. (3.4); however, unlike Eq. (2.3), the action (3.4) is necessarily (imaginary-)time reversal invariant.

Substituting Eq. (3.4) back into Eq. (3.2) and using Bose frequencies also in the cavity, one gets

$$S\{\boldsymbol{\chi}\}=Z^{-1}\int\mathcal{D}\boldsymbol{\phi}_{\rm c}\exp\left\{T\sum_m\left[(\boldsymbol{\phi}_m,\boldsymbol{\chi}_{-m})-\frac{1}{2}|\nu_m||\phi_m(a)|^2-\int_0^a dx\left(\frac{1}{2\rho}|\hat{\phi}_m|^2+\frac{1}{2}|\phi_m'|^2-\nu_m\phi_m\hat{\phi}_{-m}\right)\right]\right\}.\qquad(3.5)$$

This form completes the elimination procedure in that it manifestly involves $\boldsymbol{\phi}_{\rm c}$ only.

IV. PERFORMING THE CAVITY-FIELD INTEGRAL IN THE QNM BASIS

Substituting $\boldsymbol{\phi}_m=\Sigma_j a_{jm}\boldsymbol{f}_j$ and $\boldsymbol{\chi}_m=\Sigma_j b_{jm}\boldsymbol{f}_j$ into Eq. (3.5) [21], the "orthogonality"

$$\int_0^{a^+}dx\rho f_jf_k=\delta_{jk}-i\frac{f_j(a)f_k(a)}{\omega_j+\omega_k},\qquad(4.1)$$

which follows from Eqs. (2.7) and (2.8), leads to

$$S\{\boldsymbol{\chi}\}=Z^{-1}\int\mathcal{D}\boldsymbol{\phi}_{\rm c}\times\exp\left\{T\sum_{jm}a_{jm}\left[-\sum_k a_{k,-m}\tilde{\mathcal{S}}_{jkm}+2\omega_jb_{j,-m}\right]\right\}\qquad(4.2)$$

$$=Z^{-1}\int\mathcal{D}\boldsymbol{\phi}_{\rm c}\times\exp\left\{T\sum_{jkm}[-\overline{a}_{jm}\overline{a}_{k,-m}\tilde{\mathcal{S}}_{jkm}-b_{jm}b_{k,-m}\tilde{\eta}_{jkm}]\right\}.\qquad(4.3)$$

Here,

$$\tilde{\mathcal{S}}_{jkm}=\frac{\nu_m[\theta(m)\omega_k-\theta(-m)\omega_j]+i\omega_j\omega_k}{\omega_j+\omega_k}f_j(a)f_k(a),$$

$$\tilde{\eta}_{jkm}=\left[\frac{\theta(m)\omega_k}{i\omega_k+\nu_m}+\frac{\theta(-m)\omega_j}{i\omega_j-\nu_m}\right]\frac{f_j(a)f_k(a)}{\omega_j+\omega_k},$$

$$\overline{a}_{jm}=a_{jm}+\sum_k\frac{b_{km}\tilde{\eta}_{jk;-m}}{\omega_j},$$

and $\theta(0)\equiv\frac{1}{2}$. In Eq. (4.2), the δ_{jk} term in Eq. (4.1) has canceled by m parity. Thus, it is the second term in Eq. (4.1) which contributes; this makes Eq. (4.2) nondiagonal (cf. the double sum Σ_{jk}). The surface values $f_j(a)f_k(a)$ in $\tilde{\mathcal{S}},\tilde{\eta}$ are a measure of dissipation, since they would vanish if the field had a node also at $x=a$ [22]. Using a basis adapted to the open system, the free-cavity and damping terms in the action of Eq. (3.5) have been combined nicely in Eq. (4.2).

The nontrivial ingredient in the completion of the square (4.3) is the QNM sum rule

$$\sum_k \frac{\tilde{S}_{jkm}\tilde{\eta}_{k\ell m}}{\omega_j\omega_k} = \sum_k \frac{\tilde{\eta}_{jkm}\tilde{S}_{k\ell m}}{\omega_j\omega_k} = \sum_k \frac{f_j(a)f_k^2(a)f_\ell(a)}{(\omega_j+\omega_k)(\omega_k+\omega_\ell)} = -\delta_{j\ell}; \tag{4.4}$$

see Eq. (4.8) and further. We thus obtain the final answer for $S\{\chi\}$ in terms of the b_{jm} [23],

$$S\{\chi\} = \exp\left\{-T\sum_{jkm} b_{jm}b_{k,-m}\tilde{\eta}_{jkm}\right\}. \tag{4.5}$$

Its relation to the temperature Green function $\mathcal{G}_{jk}(\tau) = -\langle \mathcal{T}_\tau\{a_j(\tau)a_k\}\rangle$ reads [24]

$$\partial_{b_{jm}}\partial_{b_{k,-m}}S\{\chi\}|_{\chi=0} = \partial_{b_{jm}}\partial_{b_{k,-m}}|_{\chi=0}\left\langle \mathcal{T}_\tau \exp\left\{T\sum_{jm} 2\omega_j b_{jm}\int_0^\beta d\tau\, e^{-i\nu_m\tau}a_j(\tau)\right\}\right\rangle$$
$$= 4\omega_j\omega_k T^2\int_0^\beta d\tau_1 d\tau_2\, e^{i\nu_m(\tau_2-\tau_1)}\langle \mathcal{T}_\tau\{a_j(\tau_1)a_k(\tau_2)\}\rangle = -4\omega_j\omega_k T\tilde{\mathcal{G}}_{jk,-m}, \tag{4.6}$$

where it is a standard result of path integration [25] that differentiation (in our case ordinary partial differentiations with respect to the *discrete* set $\{b_{jm}\}$) of $S\{\chi\}$ automatically yields time-ordered expectation values [see below Eq. (3.2)]. Substituting Eq. (4.5) into Eq. (4.6), one obtains

$$2\omega_j\omega_k\tilde{\mathcal{G}}_{jkm} = \tilde{\eta}_{jk;-m}. \tag{4.7}$$

Quite generally, $\mathcal{G}$ is related to the real-time $G^{\mathrm{R}}_{jk}(t) = -i\theta(t)\langle[a_j(t),a_k]\rangle$ by $\tilde{\mathcal{G}}_{jkm} = \tilde{G}^{\mathrm{R}}_{jk}(i\nu_m)$ for $m \geqslant 1$ [26]. Evaluating the analytically continued $\tilde{G}^{\mathrm{R}}_{jk}$ at the frequencies $i\nu_m$, Eq. (4.7) thus is readily seen to agree exactly with the results found in Ref. [9] by canonical quantization.

Equation (4.4) will now be derived. Expanding into partial fractions if $j\neq\ell$, one sees that the sum indeed vanishes if $\Sigma_k f_k^2(a)/(\omega_k+\omega_j)$ is independent of j. This follows from

$$\sum_k \frac{f_k^2(a)}{\omega_k+\omega_j} = \sum_k \frac{f_k^2(a)}{\omega_k} + 2\omega_j\tilde{G}^{\mathrm{R}}(a,a;-\omega_j) = \sum_k \frac{f_k^2(a)}{\omega_k} + i, \tag{4.8}$$

where the last step is valid because more generally one has

$$2\omega_j\, f_j(a)\tilde{G}^{\mathrm{R}}(x,a;-\omega_j) = i f_j(x). \tag{4.9}$$

For a proof, let $\omega\to-\omega_j$ in the purely classical identity

$$\tilde{G}^{\mathrm{R}}(x,y;\omega) - \tilde{G}^{\mathrm{R}}(x,y;-\omega) = \frac{2\omega}{i}\tilde{G}^{\mathrm{R}}(x,a;\omega)\tilde{G}^{\mathrm{R}}(y,a;-\omega)$$

[9], and compare residues on both sides. The value of the first term on the RHS of Eq. (4.8) is irrelevant for the derivation of Eq. (4.4); in Appendix B it will be shown that, if $\rho(x=a)$ has a step, $\Sigma_k f_k^2(a)/(\omega_k+\omega_j) = i[\rho(a^-)+1]/[\rho(a^-)-1]$.

For $j=\ell$, we need $-\Sigma_k f_k^2(a)/(\omega_k+\omega_j)^2 = 2\partial_\omega[\omega\tilde{G}^{\mathrm{R}}(a,a;\omega)]_{\omega=-\omega_j}$. With $f(\omega)$ $[g(\omega)]$ solving Eq. (2.4) (with $\omega_j\mapsto\omega$) under the left (right) boundary condition only, one has

$$\tilde{G}^{\mathrm{R}}(x<y;\omega) = \frac{f(x,\omega)g(y,\omega)}{W(\omega)}, \tag{4.10}$$

where one can choose $f(\omega)=f(-\omega)$, and with $W=fg'-gf'$ the position-independent Wronskian of f and g [10]. Together with the OWC for $f(\omega_j)=f_j$ and g, Eq. (4.10) yields

$$4\omega_j\, f_j(a)\partial_\omega[\omega\tilde{G}^{\mathrm{R}}(a,a;\omega)]_{\omega=-\omega_j} = i\omega_j\partial_\omega\, f(a,\omega_j) + i f_j(a) - \partial_\omega\, f'(a^+,\omega_j). \tag{4.11}$$

If ω_j were a double QNM [15], i.e., if f satisfied the OWC up to $\mathcal{O}(\omega-\omega_j)$, the RHS would vanish. However, except in Appendix A we assume simple poles, so further evaluation is needed. Solve $[\partial_x^2+\rho\omega^2]\partial_\omega\, f|_{\omega_j} = -2\omega_j\rho f_j$ by varying the constant, $\partial_\omega\, f|_{\omega_j} = h_j f_j$, leading to

$$h_j'(x)f_j^2(x) = -2\omega_j\int_0^x dy\,\rho(y)f_j^2(y) \Rightarrow h_j'(a^+) = \frac{i-(\boldsymbol{f}_j,\boldsymbol{f}_j)}{f_j^2(a)} \Rightarrow i\omega_j\partial_\omega\, f(a,\omega_j) + i f_j(a) - \partial_\omega\, f'(a^+,\omega_j) = \frac{(\boldsymbol{f}_j,\boldsymbol{f}_j)}{f_j(a)};$$

since Eq. (4.11) vanishes for a double QNM one could have expected an answer $\propto(\boldsymbol{f}_j,\boldsymbol{f}_j)$ [15], which here equals $2\omega_j$. Substition into Eq. (4.11) yields

$$2\partial_\omega[\omega\tilde{G}^{\rm R}(a,a;\omega)]_{\omega=-\omega_j}=f_j^{-2}(a),\qquad(4.12)$$

completing the proof of the sum rule (4.4) and therefore of Eq. (4.3).

V. ONE-COMPONENT FORMS

The canonical analysis [9] suggests that $S\{\boldsymbol{\chi}\}$ also has a diagonal form (see Appendix C) if $\boldsymbol{\chi}$ couples only to the first component ϕ. In this section we thus set $\chi=0$, and consider $S\{\hat{\chi}\}$. Then Eq. (4.4) implies $\Sigma_k[f_j(a)f_k(a)/(\omega_j+\omega_k)]b_{km}=-ib_{jm}$, at once yielding

$$S\{\hat{\chi}\}=\exp\left\{T\sum_{jm}b_{jm}b_{j,-m}\frac{\omega_j}{\omega_j-i|\nu_m|}\right\}.\qquad(5.1)$$

However, the f_j are overcomplete for the one-component expansion of $\hat{\chi}$, so the b_{jm} are no more independent, see above Eq. (5.1). It is thus better to write out $b_{jm}=(i/2\omega_j)\int_0^{a^+}dx\, f_j\hat{\chi}_m$:

$$S\{\hat{\chi}\}=\exp\left\{-T\sum_{jm}\int_0^{a^+}dxdy\,\hat{\chi}_m(x)\hat{\chi}_{-m}(y)\times\frac{f_j(x)f_j(y)}{4\omega_j(\omega_j-i|\nu_m|)}\right\}\qquad(5.2)$$

$$=\exp\left\{T\sum_m\int_0^{a^+}dxdy\frac{1}{2}\hat{\chi}_m(x)\hat{\chi}_{-m}(y)\tilde{\mathcal{G}}_m(x,y)\right\}.\qquad(5.3)$$

While it looks simple, Eq. (5.3) [in particular the form (5.2) of $\tilde{\mathcal{G}}$] is hard to derive from Eq. (3.5) without using the power of the two-component expansion in the intermediate steps, see below Eq. (5.5). Using Eq. (5.2) for $S\{\hat{\chi}\}=\langle\mathcal{T}_\tau\exp\{i\int_0^\beta d\tau\int_0^{a^+}dx\,\hat{\chi}(x,\tau)\phi(x,\tau)\}\rangle$, ϕ-ϕ correlators follow by functional differentiation with respect to $\hat{\chi}$, without the problems of Eq. (5.1).

Also the effective cavity action has a diagonal form, possibly useful when, e.g., numerically studying an interacting extension (see Ref. [9], Secs. VI and VII). Integrating out $\hat{\phi}$ if $\chi=0$,

$$S\{\hat{\chi}\}=Z^{-1}\int\mathcal{D}\phi_{\rm c}\exp\left\{T\sum_m\left[\int_0^{a^+}dx\left(i\phi_m\hat{\chi}_{-m}-\frac{1}{2}\rho\nu_m^2|\phi_m|^2-\frac{1}{2}|\phi_m'|^2\right)-\frac{1}{2}|\nu_m||\phi_m(a)|^2\right]\right\}$$

$$=Z^{-1}\int\mathcal{D}\phi_{\rm c}\exp\left\{T\sum_{jm}a_{jm}\left[2\omega_jb_{j,-m}-\sum_k a_{k,-m}\left(\frac{\nu_m^2+\omega_j^2}{2}\delta_{jk}+i\frac{(\omega_j-i|\nu_m|)(\omega_k-i|\nu_m|)}{2(\omega_j+\omega_k)}f_j(a)f_k(a)\right)\right]\right\}.\qquad(5.4)$$

The a_{jm} are now given by $a_{jm}=\frac{1}{2}\int_0^{a^+}dx\rho\phi_m f_j+(i/2\omega_j)\phi_m(a)f_j(a)=(1/2\omega_j^2)\int_0^{a^+}dx\phi_m' f_j'$, implying $\Sigma_k[\omega_k/(\omega_j+\omega_k)]f_j(a)f_k(a)a_{km}=-i\omega_ja_{jm}$. Also using $\Sigma_j f_j^2(a)/\omega_j^2=2a$ [see Eq. (B2)] and $\phi_m(a)\phi_{-m}(a)=-i(\boldsymbol{\phi}_m,\boldsymbol{\phi}_{-m})=-2i\Sigma_j\omega_ja_{jm}a_{j,-m}$, $S\{\hat{\chi}\}$ can be written as

$$S\{\hat{\chi}\}=Z^{-1}\int\mathcal{D}\phi_{\rm c}\exp\left\{T\sum_{jm}a_{jm}[2\omega_jb_{j,-m}-a_{j,-m}\alpha_{jm}]\right\},\qquad(5.5)$$

with $\alpha_{jm}=\nu_m^2+\omega_j^2-i\omega_j(|\nu_m|+a\nu_m^2)$. The a_{jm} are no more independent [see Eq. (5.1)] as, up to $a_{jm}=a_{-j,-m}^*$, they were in Sec. IV. Thus, evaluating Eq. (5.5) directly is difficult, and Eq. (5.1) is best obtained via the auxiliary field $\hat{\phi}$ as before. One can again write out the a_{jm}:

$$S\{\hat{\chi}\}=\int\frac{\mathcal{D}\phi_{\rm c}}{Z}\exp\left\{T\sum_m\int_0^{a^+}dx\left[i\phi_m\hat{\chi}_{-m}-\int_0^{a^+}dy\,\phi_m'(x)\phi_{-m}'(y)\sum_j\frac{\alpha_{jm}f_j'(x)f_j'(y)}{4\omega_j^4}\right]\right\}.\qquad(5.6)$$

VI. DISCUSSION

As mentioned in Sec. I, this work in a sense complements [4,12] for the field models (2.1)–(2.3). Comparing Secs. IV and V clarifies why phase-space integration is essential in Eq. (3.2). Another typical feature of our dissipative system is the nontrivial step needed to proceed from Eq. (4.2) to (4.3) and thus to Eq. (4.5). Also, the analysis of Refs. [8,9] has now been extended to critically damped excitations—the Jordan-block modes of Appendix A.

By path-integral quantizing the open wave system, we have met the challenge set out in Ref. [9]. This leaves that paper's second challenge: the inclusion of matter [27]. For interactions confined to the cavity, the elimination of $\boldsymbol{\phi}_{\rm b}$ in Sec. III is not affected; one simply has an extra term in the exponent of Eq. (3.5). Section IV contains two steps (see Sec. I): the first, QNM expansion of the effective action, goes through for any interaction since the QNMs are complete. One obtains a generalization of Eq. (4.2), and any further analysis now benefits from the discrete basis. While the second step of exact evaluation will typically be impossible, by setting $a_{jm}\mapsto\partial/\partial(2\omega_jTb_{j,-m})$ in the interaction term one can write $S\{\boldsymbol{\chi}\}$ as a functional of the free $S_0\{\boldsymbol{\chi}\}$ of Eq. (4.5) [25]. With a toy action $\mathcal{S}_{\rm int}=\int_0^\beta d\tau\int_0^a dx\lambda(x)\phi^4(x,\tau)$, we get

$$S\{\boldsymbol{\chi}\}=Z^{-1}\exp\left\{-\beta\sum_{m_1\ldots m_4}\delta_{m_1+\cdots+m_4}\times\sum_{j_1\ldots j_4}\lambda_{j_1\ldots j_4}\prod_{i=1}^4\frac{1}{2\omega_{j_i}}\frac{\partial}{\partial b_{j_im_i}}\right\}S_0\{\boldsymbol{\chi}\},\qquad(6.1)$$

with $\lambda_{j_1\ldots j_4}=\int_0^a dx\lambda f_{j_1}\cdots f_{j_4}$. This form is useful when doing perturbation theory.

The final result (4.5) can also be found indirectly even without using the canonical theory: its kernel (4.7) is the analytic continuation of $\tilde{G}^{\rm R}_{jk}(\omega)$. The $G^{\rm R}_{jk}(t)$ are the unique QNM coefficients of $\mathsf{G}^{\rm R}(x,y;t)=-i\theta(t)\langle[\boldsymbol{\phi}(x,t),\boldsymbol{\phi}(y)]\rangle$, obtainable by trivial time differentiations from its (1,1) component, the latter being the classical propagator (2.5) (see Ref. [9], Secs. III and VI B, and Appendix C) [28]. Yet, the present explicit path integration is of considerable interest, since so few of them can be done in closed form unless they trivially factorize into ordinary integrals over NMs. Also, the calculation has uncovered new results on QNMs—e.g., Eq. (4.4)—which are useful already on the classical level.

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APPENDIX A: JORDAN-BLOCK PATH INTEGRAL

In the main text we have assumed that all poles in $\tilde{G}^{\rm R}(\omega)$ are simple [cf., e.g., the Fourier transform of Eq. (2.5)]. Here we study the general case, following Ref. [15] throughout [29]. For each QNM pole of order M_j in $\tilde{G}^{\rm R}(\omega)$ at $\omega=\omega_j$, introduce $f^n_j(x)=(n!)^{-1}\partial^n_\omega|_{\omega_j} f(x,\omega)$ for $0\le n\le M_j-1$, with $f(x,\omega)$ defined above Eq. (4.10). The conjugate momenta read

$$\hat{f}^n_j=-i\rho[\omega_j f^n_j+f^{n-1}_j] \tag{A1}$$

($f^n_j\equiv 0$ for $n\le -1$), so that $\boldsymbol{f}^0_j=\boldsymbol{f}_j$ is the QNM eigenvector, which together with $\{\boldsymbol{f}^n_j\}_{n=1}^{M_j-1}$ spans a so-called Jordan block of the Hamiltonian $\mathcal{H}$ of Sec. II.

One main result of Ref. [15] now reads: if one chooses f and g such that, for all j,

$$g(x,\omega)=f(x,\omega)+\mathcal{O}[(\omega-\omega_j)^{M_j}],$$
$$W(\omega)=2\omega_j(\omega-\omega_j)^{M_j}+\mathcal{O}[(\omega-\omega_j)^{2M_j}] \tag{A2}$$

(note the orders of the errors), which is readily achieved, one has the biorthogonality relation

$$(\boldsymbol{f}^n_j,\boldsymbol{f}^r_k)=2\omega_j\delta_{jk}\delta_{n+r,M_j-1}\quad(0\le n\le M_j-1, 0\le r\le M_k-1). \tag{A3}$$

Expanding $\boldsymbol{\phi}_m=\Sigma'_{jn}a^n_{jm}\boldsymbol{f}^n_j$ and $\boldsymbol{\chi}_m=\Sigma'_{jn}b^n_{jm}\boldsymbol{f}^n_j$ ($\Sigma'_n\equiv\Sigma_{n=0}^{M_j-1}$) in Eq. (3.5), the term with $f^{n\prime}_j$ can again be integrated by parts, using $[\partial_x^2+\rho\omega_j^2]f^n_j=-\rho[2\omega_j f^{n-1}_j+f^{n-2}_j]$ and the OWC $f^{n\prime}_j(a^+)=-\hat{f}^n_j(a^+)$. In general, $\int_0^{a^+}dx\rho f^n_j f^r_k$ is not reduced to surface terms by Eq. (A3) in one step as in Eq. (4.1), because of the second term in Eq. (A1); iteration leads to

$$\int_0^{a^+}dx\rho f^n_j f^r_k=\frac{\delta_{jk}\theta(n+r+\frac{3}{2}-M_j)}{(-\omega_j)^{n+r+1-M_j}}+i\sum_{p=0}^{n}\sum_{q=0}^{r}\binom{p+q}{p}\frac{f^{n-p}_j(a)f^{r-q}_k(a)}{(-\omega_j-\omega_k)^{p+q+1}} \tag{A4}$$

$$=\frac{\delta_{jk}\theta\left(n+r+\frac{3}{2}-M_j\right)}{(-\omega_j)^{n+r+1-M_j}}-i\frac{\partial^n_\omega|_{\omega_j}}{n!}\frac{\partial^r_\mu|_{\omega_k}}{r!}\frac{f(a,\omega)f(a,\mu)}{\omega+\mu}, \tag{A5}$$

where the definition of f^n_j above Eq. (A1) shows that Eq. (A5) equals Eq. (A4). Compact forms such as Eq. (A5) will be essential below. Comparing the second terms on their respective RHSs, Eq. (A5) is seen to be a differentiated version of Eq. (4.1). However, Eq. (A5) depends on Eq. (A2). One obtains

$$S\{\boldsymbol{\chi}\}=Z^{-1}\int\mathcal{D}\boldsymbol{\phi}_{\rm c}\exp\left\{T\sum_{jnm}{}' a^n_{jm}\left[-\sum_{kr}{}' a^r_{k,-m}\tilde{\mathcal{S}}^{nr}_{jkm}+2\omega_j b^{M_j-1-n}_{j,-m}\right]\right\}, \tag{A6}$$

$$\tilde{\mathcal{S}}^{nr}_{jkm}=\frac{\partial^n_\omega|_{\omega_j}}{n!}\frac{\partial^r_\mu|_{\omega_k}}{r!}f(a,\omega)f(a,\mu)\times\frac{\nu_m[\mu\theta(m)-\omega\theta(-m)]+i\omega\mu}{\omega+\mu}. \tag{A7}$$

The first term of Eq. (A5) cancels in $\tilde{\mathcal{S}}$, see below Eq. (4.3). We claim that the result of Eq. (A6) is

$$S\{\boldsymbol{\chi}\}=\exp\left\{-2T\sum_{jknrm}{}'\omega_j\omega_k b^n_{jm}b^r_{k,-m}\tilde{\mathcal{G}}^{M_j-1-n,M_k-1-r}_{jk;-m}\right\}, \tag{A8}$$

$$\tilde{\mathcal{G}}^{nr}_{jkm}=\frac{\partial^{M_j-1-n}_\lambda|_{\omega_j}}{(M_j-1-n)!}\frac{\partial^{M_k-1-r}_\mu|_{\omega_k}}{(M_k-1-r)!}\frac{f(a,\lambda)f(a,\mu)}{2\omega_j\omega_k(\lambda+\mu)}\times\left\{\frac{\theta(m)\lambda}{i\lambda+\nu_m}+\frac{\theta(-m)\mu}{i\mu-\nu_m}\right\}. \tag{A9}$$

To verify this claim, it suffices to show that [23] $U^{nu}_{j\ell m}\equiv\Sigma'_{kr}\tilde{\mathcal{S}}^{nr}_{jkm}\tilde{\mathcal{G}}^{ur}_{\ell km}=-\frac{1}{2}\delta_{j\ell}\delta_{nu}=\Sigma'_{kr}\tilde{\mathcal{S}}^{rn}_{kjm}\tilde{\mathcal{G}}^{ru}_{k\ell m}$; since $\tilde{\mathcal{S}}^{nr}_{jkm}=\tilde{\mathcal{S}}^{rn}_{kj;-m}$ and $\tilde{\mathcal{G}}^{nr}_{jkm}=\tilde{\mathcal{G}}^{rn}_{kj;-m}$, these two relations are equivalent. To evaluate $U^{nu}_{j\ell m}$, first for $\nu_m>0$, substitute Eqs. (A7) and (A9), doing Σ'_r by the product rule:

$$U^{nu}_{j\ell m}=\frac{\partial^n_\omega|_{\omega_j}}{n!}\frac{\partial^{M_\ell-1-u}_\lambda|_{\omega_\ell}}{(M_\ell-1-u)!}\frac{\nu_m+i\omega}{\nu_m+i\lambda}\frac{\lambda}{\omega_\ell}\times f(a,\omega)f(a,\lambda)\sum_k\frac{\partial^{M_k-1}_\mu|_{\omega_k}}{(M_k-1)!}\frac{\mu f(a,\mu)^2}{2\omega_k(\mu+\omega)(\mu+\lambda)}. \tag{A10}$$

Rewriting $\mu/[(\mu+\omega)(\mu+\lambda)]=[\omega/(\mu+\omega)-\lambda/(\mu+\lambda)]/(\omega-\lambda)$ in Eq. (A10), one recognizes [15]

$$\tilde{G}^{\mathrm{R}}(x,y;\zeta)=\sum_k \frac{\partial_\mu^{M_k-1}|_{\omega_k}}{(M_k-1)!}\frac{f(x,\mu)f(y,\mu)}{2\omega_k(\zeta-\mu)}. \quad \text{(A11)}$$

Replace Eq. (A11) by Eq. (4.10), and in the latter use $f(\omega)=f(-\omega)$ to write $\tilde{G}^{\mathrm{R}}(x,a;-\omega)=-f(x,\omega)/[f'(a^+,\omega)+i\omega f(a,\omega)]$. With $f'(a^+,\omega)=i\omega f(a,\omega)-2\omega_j(\omega-\omega_j)^{M_j}/f(a,\omega)+\mathcal{O}[(\omega-\omega_j)^{2M_j}]$ [see Eq. (A2)], this leads to

$$\omega\tilde{G}^{\mathrm{R}}(x,a;-\omega)=\frac{if(x,\omega)}{2f(a,\omega)}+\frac{\omega_j f(x,\omega)(\omega-\omega_j)^{M_j}}{2\omega f^3(a,\omega)}+\mathcal{O}[(\omega-\omega_j)^{2M_j}], \quad \text{(A12)}$$

generalizing both Eqs. (4.9) and (4.12). The error term of Eq. (A12) does not contribute in Eq. (A14) below, and from now on will be omitted. Substituting Eq. (A12) [with $x\mapsto a$, and $(\omega,j)\mapsto(\lambda,\ell)$ in the second partial fraction above Eq. (A11)] into the upshot of Eq. (A10), one finds

$$U^{nu}_{j\ell m}=\frac{\partial_\omega^n|_{\omega_j}}{n!}\frac{\partial_\lambda^{M_\ell-1-u}|_{\omega_\ell}}{(M_\ell-1-u)!}\times\frac{\nu_m+i\omega}{\nu_m+i\lambda}\frac{\lambda}{\omega_\ell}\frac{f(a,\omega)f(a,\lambda)}{\omega-\lambda}\times\left[\frac{\omega_\ell(\lambda-\omega_\ell)^{M_\ell}}{2\lambda f^2(a,\lambda)}-\frac{\omega_j(\omega-\omega_j)^{M_j}}{2\omega f^2(a,\omega)}\right]. \quad \text{(A13)}$$

This vanishes if $j\neq\ell$, since then $(\omega-\lambda)^{-1}$ is regular while $n\leqslant M_j-1$ and $u\geqslant 0$. If $j=\ell$, in the first term of Eq. (A13) use $(\nu_m+i\omega)f(a,\omega)/[(\nu_m+i\lambda)f(a,\lambda)]=1+\mathcal{O}(\omega-\lambda)$; the error term does not contribute as it cancels $(\omega-\lambda)^{-1}$, upon which $(\lambda-\omega_j)^{M_j}$ yields zero in the final differentiation. Handling the second term of Eq. (A13) analogously, one is left with

$$\begin{aligned}U^{nu}_{jjm}&=\frac{\partial_\omega^n|_{\omega_j}}{n!}\frac{\partial_\lambda^{M_j-1-u}|_{\omega_j}}{(M_j-1-u)!}\frac{(\lambda-\omega_j)^{M_j}-(\omega-\omega_j)^{M_j}}{2(\omega-\lambda)}\\&=-\frac{1}{2}\frac{\partial_\omega^n|_{\omega_j}}{n!}\frac{\partial_\lambda^{M_j-1-u}|_{\omega_j}}{(M_j-1-u)!}\times\sum_p{}'\,(\omega-\omega_j)^p(\lambda-\omega_j)^{M_j-1-p}\\&=-\frac{1}{2}\delta_{nu},\end{aligned} \quad \text{(A14)}$$

proving our claim for $\nu_m>0$; $\nu_m\leqslant 0$ is similar, but factors with ν_m cancel from the outset.

For a check on the algebra and a closer look at the unusual Jordan-block excitations, comparison with canonical quantization is instructive. Expand the Heisenberg field $\boldsymbol{\phi}_{\mathrm{c}}(t)=\Sigma'_{jn}a_j^n(t)\boldsymbol{f}_j^n$; the operators a_j^n satisfy a coupled (for different n) system of Langevin equations [8,9], from which they may be solved in terms of the thermal and quantum noise incoming from the outside, having a simple Planck distribution (we omit the details). Since $G^{\mathrm{R}nr}_{jk}(t)=-i\theta(t)\langle[a_j^n(t),a_k^r]\rangle$ turns out to be related to the $\tilde{\mathcal{G}}$ of Eq. (A9) as below Eq. (4.7), the path integral and canonical approaches indeed agree for arbitrary QNM pole configurations.

APPENDIX B: QNM SUM RULES

In Ref. [10], $\Sigma_j\, f_j(x)f_j(y)/\omega_j=0$ ($\Sigma_j\equiv\lim_{M\to\infty}\Sigma_{j=-M}^M$) follows from $G^{\mathrm{R}}(x,y;t=0)=0$ in Eq. (2.5). However, pointwise this only holds if $x\neq y$. Setting $\rho(x)=\rho_{\mathrm{f}}(x)+\mu\delta(x-a)$ with finite ρ_{f}, countour integration of $\tilde{G}^{\mathrm{R}}(\omega)$ in the upper half ω-plane (only the large semicircle contributes, on which one can use WKB methods [30]) gives $G^{\mathrm{R}}(x,x;0)=-\frac{1}{2}[n(x^-)+n(x^+)]^{-1}$ ($n\equiv\sqrt{\rho}$) if $x<a$ or $\mu=0$, while $G^{\mathrm{R}}(a,a;0)=0$ if $\mu>0$. This agrees with a real-time analysis showing that $G^{\mathrm{R}}(x,x;0^+)=-[n(x^-)+n(x^+)]^{-1}$ [$G^{\mathrm{R}}(a,a;0^+)=0$, $\mu>0$] while of course $G^{\mathrm{R}}(t=0^-)=0$. Now integrate $\tilde{G}^{\mathrm{R}}$ in the lower half plane; comparison yields

$$\sum_j \frac{f_j(x)f_j(x)}{\omega_j}=\begin{cases}0, & 0\leqslant x<a \text{ or } \mu>0,\\ 2i/[\rho(a^-)-1], & x=a,\mu=0,\end{cases} \quad \text{(B1)}$$

leading to the result for Eq. (4.8) quoted below Eq. (4.9) if $\mu=0$; for $\mu>0,\Sigma_k f_k^2(a)/(\omega_k+\omega_j)=i$.

For another sum, solve the trivial differential equation for $\tilde{G}^{\mathrm{R}}(\omega=0)$. One finds

$$\sum_j \frac{f_j(x)f_j(y)}{\omega_j^2}=-2\tilde{G}^{\mathrm{R}}(x,y;0)=2\min(x,y) \quad (0\leqslant x,y\leqslant a). \quad \text{(B2)}$$

Clearly, ∂_x^2(B2) reproduces $\rho(x)\Sigma_j\, f_j(x)f_j(y)=2\delta(x-y)$ for $0<x,y<a$ [10]. For $\rho(x<a)=\text{const}$, Eqs. (B1) and (B2) become conventional Fourier series [9].

APPENDIX C: DIAGONAL TWO-VARIABLE QNM EXPANSION

Motivated by "diagonal" series such as Eqs. (5.2) and (5.6) (also found in Ref. [9]), we study the *one-component* expansion

$$\phi(x,y)=\sum_j a_j\, f_j(x)f_j(y) \quad \text{(C1)}$$

for symmetric ϕ: $[0,a]^2 \to \mathbf{C}$. Since $[\rho^{-1}(x)\partial_x^2 - \rho^{-1}(y)\partial_y^2]\phi(x,y)=0$, ϕ is determined by a set of boundary conditions specifying a unique solution to a hyperbolic equation. Hence, $\{f_j(x)f_j(y)\}$ is grossly undercomplete in the space of functions on $0 \leqslant x \leqslant y \leqslant a$. However, the expansion (C1) is unique if $\Sigma_j a_j$ converges absolutely. For a proof, suppose $\phi=0$, i.e.,

$$\sum_j a_j f_j(x) f_j(y) = 0. \tag{C2}$$

Operating on (C2) with $\int_0^{a^+} dx\, \rho(x) f_k(x)$ shows that $[\gamma_{jk} \equiv \delta_{jk} - i f_j(a) f_k(a)/(\omega_j + \omega_k)]$

$$\sum_j a_j f_j(y)\gamma_{jk} = 0, \tag{C3}$$

while $-i\partial_y(\text{C3})|_{a^-}$ yields

$$\sum_j a_j(\omega_j - i\mu\omega_j^2) f_j(a)\gamma_{jk} = 0 \tag{C4}$$

(see Appendix B for μ) and $-\rho^{-1}(a^-)\partial_y^2(\text{C3})|_{a^-}$ gives

$$\sum_j a_j \omega_j^2 f_j(a)\gamma_{jk} = 0. \tag{C5}$$

Finally, $\omega_k(\text{C3})|_{y=a} + (\text{C4}) + i\mu(\text{C5}) + i f_k(a)(\text{C2})_{x=y=a}$ reads

$$2\omega_k f_k(a) a_k = 0 \;\Rightarrow\; a_k = 0. \quad \square \tag{C6}$$

The summability of $\{a_j\}$ enables taking $y \uparrow a$ behind Σ_j to arrive at Eqs. (C4)–(C6): for $\mu>0, \gamma_{jk} = \mathcal{O}(j^{-2})$ and this is multiplied at most with $a_j\omega_j^2 f_j(y)$ where $f_j(y)$ is bounded. Hence, Σ_j converges uniformly with respect to y. If $\mu=0$, $\gamma_{jk} = \mathcal{O}(j^{-1})$, but now the prefactor is at most $a_j\omega_j f_j(y)$ since Eq. (C5) is not needed. While we have not exhaustively examined slowly converging or distributional series (C1), the above suggests strongly that the only freedom then is the addition of c/ω_j to a_j if $\mu>0$; since we also supposed convergence *at* $x=y=a$, by Eq. (B1) even this freedom is absent if $\mu=0$ (step discontinuity).

[1] R. Lang, M.O. Scully, and W.E. Lamb, Phys. Rev. A **7**, 1788 (1973); R. Lang and M.O. Scully, Opt. Commun. **9**, 331 (1973); H. Dekker, *ibid.* **10**, 114 (1974); Physica C **83**, 183 (1976); A.J. Campillo, J.D. Eversole, and H.-B. Lin, Phys. Rev. Lett. **67**, 437 (1991).

[2] P. Meystre and M. Sargent III, *Elements of Quantum Optics* (Springer, Berlin, 1991).

[3] G. Schön and A.D. Zaikin, Phys. Rep. **198**, 237 (1990).

[4] U. Weiss, *Quantum Dissipative Systems* (World Scientific, Singapore, 1993).

[5] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, Oxford, 1983).

[6] E.S.C. Ching, P.T. Leung, W.M. Suen, and K. Young, Phys. Rev. D **54**, 3778 (1996).

[7] H. Dekker, Phys. Rep. **80**, 1 (1981).

[8] P.T. Leung, A. Maassen van den Brink, and K. Young, in *Frontiers in Quantum Physics*, proceedings of the International Conference, Kuala Lumpur, edited by S. C. Lim, R. Abd-Shukor, and K. H. Kwek (Springer, Singapore, 1998), p. 214.

[9] K.C. Ho, P.T. Leung, A. Maassen van den Brink, and K. Young, Phys. Rev. E **58**, 2965 (1998).

[10] E.S.C. Ching, P.T. Leung, A. Maassen van den Brink, W.M. Suen, S.S. Tong, and K. Young, Rev. Mod. Phys. **70**, 1545 (1998).

[11] R.W.F. van der Plank and L.G. Suttorp, Phys. Rev. A **53**, 1791 (1996).

[12] R.P. Feynman and F.L. Vernon, Ann. Phys. (N.Y.) **24**, 118 (1963); A.O. Caldeira and A.J. Leggett, *ibid.* **149**, 374 (1983).

[13] Since $\rho>0$ limits the singularity of $\rho(x=a)$ to at most a δ function, ϕ itself is continuous. Stronger singularities would also leave equations such as Eq. (2.4) undefined distributionally.

[14] A. Maassen van den Brink and K. Young (unpublished).

[15] A. Maassen van den Brink and K. Young, e-print math-ph/9905019.

[16] Since the Euclidean phase-space action does not contain a term $\propto \phi^2$ in the absence of mass regularization, there is no Gaussian ϕ cutoff and the source term should be purely imaginary if the path integral is to have a fighting chance of being definable. Due to the overall factor i in the bilinear map, the choice in Eq. (3.2) satisfies this condition for real $\boldsymbol{\chi}$.

[17] L.S. Schulman, *Techniques and Applications of Path Integration* (Wiley, New York, 1981).

[18] G. Roepstorff, *Path Integral Approach to Quantum Physics* (Springer, Berlin, 1994).

[19] When including interactions as contemplated in Sec. VI, phase-space integration should not lead to problems, since at least in the perturbative regime the interacting path integral merely is a formal tool for arriving at the diagram expansion.

[20] E.g., H. Dekker, Phys. Lett. **104A**, 72 (1984).

[21] The QNM expansion is thus applied to all two-component fields, since the path integral is not restricted to only outgoing ones, see below Eq. (3.4). For a justification, see Refs. [9] (where one faces the analogous issue for operator fields) and [14].

[22] However, from this one may not incorrectly conclude that the effective action vanishes in the conservative limit. Rather, if $j=-k$ also $\omega_j + \omega_k \to 0$ in the action of Eq. (4.2); in the limit the diagonal contribution of these terms only yields the action of the closed cavity in terms of its NMs. See also Ref. [9], the end of Sec. VII, and Appendix A.

[23] A subtlety in Eq. (4.3) is that $\overline{a}_{jm} \neq \overline{a}^*_{-j,-m}$ in general, making the integration in $\overline{a}$ space b dependent. However, one can verify (most systematically by splitting the $\overline{a}$ integrals into real and imaginary parts and subsequently using contour methods) that this does *not* affect the result: the full b dependence in Eq. (4.3) is the one indicated explicitly.

[24] With the standard shorthand $\partial_b \equiv \frac{1}{2}[\partial_{\mathrm{Re}\, b} - i\partial_{\mathrm{Im}\, b}]$ (and $\partial_{b*} \equiv \frac{1}{2}[\partial_{\mathrm{Re}\, b} + i\partial_{\mathrm{Im}\, b}]$), one can use the b_{jm}, obeying b_{jm}

$=b^{*}_{-j,-m}$, formally as if they were independent variables.

[25] L.H. Ryder, *Quantum Field Theory* (Cambridge University Press, Cambridge, 1986).

[26] A.A. Abrikosov, L.P. Gor'kov, and I.E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Dover, New York, 1975).

[27] K.C. Ho, P.T. Leung, A. Maassen van den Brink, and K. Young, in *Proceedings of the APPC7 Conference*, edited by H.S. Chen (Science Press, Beijing, 1999), p. 433.

[28] To streamline a lengthy argument in Ref. [9], set $\tilde{\mathsf{F}}(x,y,\omega) \equiv \langle \tilde{\boldsymbol{\phi}}(x,\omega) \otimes \boldsymbol{\phi}(y) \rangle$ as in Eq. (6.6) (all cross references are in Ref. [9]). A key point of Ref. [9] is that $\boldsymbol{\phi}$ has a QNM expansion (2.8) just as its classical counterpart, so $\tilde{\mathsf{F}}(x,y,\omega) = \Sigma_{jk} \langle \tilde{a}_j(\omega) a_k \rangle \boldsymbol{f}_j(x) \otimes \boldsymbol{f}_k(y)$. Equations. (6.2), (6.3), and (6.7) now at once yield Eq. (6.8) for $\tilde{a}_{jk}$. While this calculation no longer uses Appendix C, the tensor expansion presented there remains useful for reference.

[29] The G^{R} here and G of Ref. [15] have opposite signs, since the usual sign for the quantum propagator is not convenient in classical field theory. The signs of $W(\omega)$ also differ.

[30] P.T. Leung, S.Y. Liu, and K. Young, Phys. Rev. A **49**, 3057 (1994).